Topics on Quadratic Forms

Alexander Vishik[*]

School of Mathematical Sciences, University of Nottingham,
University Park, Nottingham, United Kingdom

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[*]alexander.vishik@nottingham.ac.uk

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Introduction

These are notes of lectures given at the "School on Algebraic K-theory and its Applications" at the Abdus Salam International Centre for Theoretical Physics, Trieste in May 2007. In these lectures I tried to give some idea about the modern state of the theory of quadratic forms, and to outline its connections to K-theory. Here K-theory is understood in a broader sense. It includes such classical parts as Milnor's K-theory as well as new areas related to Motivic Homotopic Topology. From the point of view of the latter, algebraic K-theory provides an example of the *generalised cohomology theory* on the category of algebraic varieties. It is related to other theories of the same sort, and, in particular, to the universal one among them - the Algebraic Cobordism theory.

It appears that the connection of quadratic forms to K-theory discovered long ago by Milnor is just the reflection of the fact that these objects describe homology and homotopy groups of a point in the motivic world, and are really basic for the Motivic Homotopic category. In turn, the motivic methods can be used to obtain information on a particular quadratic form. Here one studies geometric properties of the canonical homogeneous varieties associated to a given form by using Chow groups and Algebraic Cobordism theory. The applications include computations of the classical invariants of quadratic forms, as well as that of the related u-invariant of fields. I will touch all these subjects, and also will try to briefly introduce the reader to Chow groups, motives, Algebraic Cobordism theory and cohomological operations in the latter.

Lecture 1

Quadratic forms and their invariants

Let k be a field of characteristic different from 2.

Let V be some finite dimensional vector space over k. Quadratic form on V is a map $q : V \to k$ which is a diagonal part of some symmetric bilinear form $B_q : V_q \times V_q \to k$. That is, $q(v) = B_q(v, v)$. It is easy to see that under our characteristic assumption B_q can be reconstructed from q uniquely.

The form is called *nondegenerate* if the respective symmetric bilinear form is, in other words, if no vector in V is orthogonal to the whole V: $V^{\perp} = 0$.

Under our assumptions, each quadratic form is *diagonalisable*, that is, one can choose the coordinates $x_1, \ldots, x_n$ on V so that $q((x_1, \ldots, x_n)) = a_1x_1^2 + \ldots + a_nx_n^2$ for certain $a_1, \ldots, a_n \in k^*$. We will denote such form $\langle a_1, \ldots, a_n\rangle$, and sometimes will call a_i-the *eigenvalues*.

Warning: in the contrast to the case of linear transformation, these "eigenvalues" are not defined uniquely, so in some other orthogonal coordinates the same form can be presented by $\langle b_1, \ldots, b_n\rangle$ for completely different set $b_1, \ldots, b_n \in k^*$. Try this on the example $\langle 1, -1\rangle$ and $\langle a, -a\rangle$, where $a \in k^*$ (hint: show that both of them are isomorphic to the form xy).

On the set of quadratic forms we have two operations: $+$ and $\cdot$

$(q_1, V_1) + (q_2, V_2) := (q_1 \perp q_2, V_1 \oplus V_2)$, where $(q_1 \perp q_2)((v_1, v_2)) = q_1(v_1) + q_2(v_2)$, and

$(q_1, V_1) \cdot (q_2, V_2) := (q_1 \otimes q_2, V_1 \otimes V_2)$, where $(q_1 \otimes q_2)(v_1 \otimes v_2) = q_1(v_1) \cdot q_2(v_2)$.

Definition 0.1 *Define* $\widetilde{W}(k)$ *- the* Grothendieck-Witt ring *of* k *as the Grothendieck group (*group completion*) of the monoid of isomorphism classes of non-degenerate quadratic forms over* k *with respect to operation* $+$*. Notice, that the operations* $+$ *and* $\cdot$ *naturally descend to* $\widetilde{W}(k)$ *and supply it with the structure of the commutative ring.*

Why study quadratic forms?

Let me give you several reasons why quadratic forms can be interesting.

1) Connected to K-theory.

More precisely, to Milnors K-theory and motivic cohomology.

Consider form $\mathbb{H} =< 1, -1 >$ called *elementary hyperbolic form*. It is an easy observation, that for arbitrary quadratic form q, $\mathbb{H} \cdot q = \mathbb{H} \perp \ldots \perp \mathbb{H}$ (the number of copies $= dim(q)$). Thus the image of the map $\mathbb{Z} \cdot \mathbb{H} \to \widetilde{W}(k)$ is an ideal in $\widetilde{W}(k)$.

Definition 0.2 *Define* $W(k)$ *- the* Witt ring *of* k *as the quotient* $\widetilde{W}(k)/\mathbb{Z}\cdot\mathbb{H}$.

Inside $W(k)$ one has the ideal I of even-dimensional forms (notice that the dimension modulo 2 is well-defined on $W(k)$). This ideal gives rise to the multiplicative filtration

$$W(k) \supset I \supset I^2 \supset I^3 \supset \ldots,$$

and one can consider the associated graded ring

$$gr_{I^\bullet}(W(k)) := W(k)/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots .$$

This ring is basically of "the same size" as $W(k)$, but with the operations $+$ and $\cdot$ somewhat damaged (some information is lost).

Milnor "Conjecture" on quadratic forms relates our graded ring with the ring called *Milnor K-theory*, where the latter is defined as follows. Consider k^* as an abelian group = $\mathbb{Z}$-module. Let $\mathrm{T}_{\mathbb{Z}}(k^*)$ be the tensor algebra of this module over $\mathbb{Z}$, that is:

$$\mathrm{T}_{\mathbb{Z}}(k^*) = \mathbb{Z} \oplus (k^*) \oplus (k^* \otimes_{\mathbb{Z}} k^*) \oplus (k^* \otimes_{\mathbb{Z}} k^* \otimes_{\mathbb{Z}} k^*) \oplus \dots .$$

Definition 0.3 *Milnor K-theory of k is defined as a quotient of the tensor algebra above by the explicit quadratic relations:*

$$\mathrm{K}^M_*(k) := \mathrm{T}_{\mathbb{Z}}(k^*)/(a \otimes (1-a), a \in k^* \backslash 1).$$

Milnor conjecture on quadratic forms states that $\mathrm{K}^M_*(k)/2$ is naturally isomorphic to $gr_{I^\bullet}(W(k))$. And Milnor K-theory is a particular case of *motivic cohomology*. So, our ring can be also interpreted as $\oplus_n \mathrm{H}^{n,n}_{\mathcal{M}}(\mathrm{Spec}(k), \mathbb{Z}/2)$ (notice, that in algebraic geometry, in contrast to topology, the cohomology are numbered by two integers, as opposed to one). If one uses also *Beilinson-Lichtenbaum "Conjecture"* (which follows from the Milnors one, and so is settled), one can see that the knowledge of quadratic forms over k gives one the complete knowledge of motivic cohomology of a point with $\mathbb{Z}/2$-coefficients.

2) Related to stable homotopy groups of spheres. In a sense, it is just the sharpened version of the reason 1).

One of the most important questions in topology (central to the mathematics as a whole) is the study of *stable homotopy groups of spheres*. Homotopy groups of spheres $\pi_n(S^m)$ count the continuous maps $S^n \to S^m$ up to homotopy (two maps are called homotopic, if you can continuously "pull" one into the other). There is a *suspension operation* Σ such that $\Sigma(S^n) = S^{n+1}$; being a functor, it acts also on the homotopy classes of maps and provides a group homomorphism $\pi_n(S^n) \to \pi_{n+1}(S^{m+1})$. The *stable homotopy groups* are defined as

$$\pi^s_n(S^0) := \lim_{N\to\infty} \pi_{n+N}(S^N).$$

Computation of these groups was performed only for small number of n.

In algebraic geometry both homotopy and homology groups are numerated by two integers (the world here is more complicated - there are two suspensions).

It was proven by F. Morel that the Grothendieck-Witt ring of quadratic forms over k describes the $(0,0)$ stable homotopy group of spheres:

$$\widetilde{W}(k) \overset{naturally}{\cong} \pi^s_{0,0}(S^0).$$

So, studying quadratic forms we study the homotopy groups of spheres, and the experience obtained here in the end could prove useful back in the topological world.

3) Quadrics give examples of *homogeneous varieties*.

To each quadratic form q one can assign the respective projective quadric $Q \subset \mathbb{P}(V_q)$ given by the equation $q = 0$. If q is nondegenerate, the respective quadratic hypersurface will be *smooth* (no singularities). The group of orthogonal linear transformation preserving the form q (denoted $O(q)$) acts naturally on Q, and the action is *transitive* in certain sense. Thus, Q is a *projective homogeneous variety* for the algebraic group $O(q)$. Other homogeneous varieties for other algebraic groups behave in many respects similar to the ones for the orthogonal group. Hence, studying the quadrics we get certain insight into the behaviour of other homogeneous varieties. Useful to mention, that all such varieties are somewhat trivial over algebraically closed field, and so here we are dealing with the pure extract of the effects which distinguish arbitrary field from the algebraically closed one (can be then used to extend the results on some other more complicated varieties from the case of algebraically closed field to that of arbitrary one).

Connection to K-theory

If you just have some arbitrary form at your disposal it is not very easy to see much K-theory in it. But some forms are better than others, and with the good forms the connection is well-visible. The best such forms are *Pfister forms*.

Pfister forms

Definition 0.4 *Let $a \in k^*$. The 1-fold Pfister form $\langle\langle a\rangle\rangle$ is the 2-dimensional form $\langle 1, -a\rangle$.*

If now $a_1, \ldots, a_n \in k^*$, then n-fold Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is the product $\langle\langle a_1\rangle\rangle \otimes \ldots \otimes \langle\langle a_n\rangle\rangle$.

Examples:

$n = 1$ $\langle\langle a\rangle\rangle = \mathrm{Nrm}_{k\sqrt{a}/k}$ - the norm map from the the quadratic extension.

$n = 2$ $\langle\langle a, b\rangle\rangle = \mathrm{Nrd}_{Quat(\{a,b\},k)/k}$ - the *reduced norm* map in the *Quaternion algebra*.

$n = 3$ $\langle\langle a, b, c\rangle\rangle = \mathrm{Nrd}_{\mathbb{O}(\{a,b,c\},k)/k}$ - the *reduced norm* map in the *Octonion algebra*.

In all three cases we have an algebra structure on the underlying vector space of quadratic form, that is a bilinear operation $* : V \times V \to V$ such that

$$q(x * y) = q(x) \cdot q(y)$$

(although, for $n = 2$ the operation is not commutative, and for $n = 3$ not even associative).

For $n > 3$ it is still possible to define such an operation $*$, but it will not be bilinear, but only linear in the 1-st coordinate, and rational in the 2-nd. And Pfister forms are the only forms for which such multiplicativity holds (if you demand this property not just over k but also over all extensions F/k).

The quadratic form q is called *isotropic* if it represents zero nontrivially (that is, there is $v \neq 0$, such that $q(v) = 0$). This property is equivalent to the fact that $\mathbb{H}$ is a direct summand in our form: $q = \mathbb{H} \perp q'$. For each quadratic form q there is unique anisotropic form q_{an} such that $q = \mathbb{H} \perp \ldots \perp \mathbb{H} \perp q_{an}$, and the number of hyperbolic summands $i_W(q)$ is called the *Witt index* (of course, it is also uniquely determined). The forms for which $dim(q_{an}) \leqslant 1$ (almost nothing left) are called *completely split*. Notice that the form is anisotropic if and only if the respective projective quadric Q has no k-rational points at all.

The <u>Main Property</u> of Pfister forms is:

$$\text{Pfister form is isotropic} \;\Leftrightarrow\; \text{it is completely split}$$

Sometimes two sets of parameters $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ define the same (isomorphic) Pfister form. It appears that this happens iff there is an equality of the respective *pure symbols* $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$ as elements of $\mathrm{K}_n^M(k)/2$ (pure symbol $\{c_1, \ldots, c_m\}$ is just the product $\{c_1\} \cdot \ldots \cdot \{c_m\}$ of elements of degree 1 in $\mathrm{K}_*^M(k)$, where $\mathrm{K}_1^M(k)$ is naturally identified with k^*).

Thus, the Pfister form depends only on pure symbol (which is also reconstructed from the form uniquely), and we can denote it as $\langle\langle\alpha\rangle\rangle$, for pure symbol $\alpha \in \mathrm{K}_n^M(k)/2$.

The *Milnor map* in the isomorphism from the Milnor "Conjecture"

$$\mathrm{K}_*^M(k)/2 \stackrel{\phi}{\to} gr_{I^\bullet}(W(k))$$

is defined as identity on 0-degree component (isomorphic to $\mathbb{Z}/2$), is given by $\phi(\{a\}) = \langle\langle a\rangle\rangle(mod. I^2)$ on the component of degree 1, and then uniquely extended as a homomorphism of algebras (the left algebra is generated by the first degree component, and it is not difficult to see that ϕ respects our explicit quadratic relations $a \otimes (1-a)$). Thus under the Milnor map the pure symbols goes to the respective Pfister forms (modulo I^{n+1}).

In a meantime, we observe that to each Pfister form we can assign two invariants:

$$\text{foldness} = n, \ \text{ and pure symbol } \ \alpha \in \mathrm{K}_n^M(k)/2,$$

from which the form itself can be reconstructed.

But Pfister forms live only in dimensions of the type 2^n. What about other dimensions? In any dimension there is a "substitute" for the Pfister form, which, may be, not as good as the Pfister form itself, but still is the best thing one can find there. These are so-called *excellent forms*. To construct an excellent form of dimension d, one has to start by presenting d in the form $2^{r_1} - 2^{r_2} + 2^{r_3} - \ldots \pm 2^{r_s}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geqslant 1$ (one can easily check that there is $1-1$ correspondence between $\mathbb{N}$ and such sequences). then for each $1 \leqslant i \leqslant s$ one has to choose pure symbol $\alpha_i \in \mathrm{K}_{r_i}^M(k)/2$ in such a way that α_s divides α_{s-1} divides ... divides α_1. Notice that $\beta = \{b_1, \ldots, b_l\}$ divides $\alpha = \{a_1, \ldots, a_m\}$ in $\mathrm{K}_*^M(k)/2$ if and only if our symbols have other presentations: $\beta = \{c_1, \ldots, c_l\}$ and $\alpha = \{c_1, \ldots, c_l, d_{l+1}, \ldots, d_m\}$.

In particular, if β divides α, then $\langle\langle\beta\rangle\rangle$ is naturally a subform of $\langle\langle\alpha\rangle\rangle$ (since $\langle 1\rangle$ is a subform of $\langle\langle d_{l+1}, \ldots, d_m\rangle\rangle$). In particular, in our situation, $\langle\langle\alpha_1\rangle\rangle \supset \langle\langle\alpha_2\rangle\rangle \supset \ldots \supset \langle\langle\alpha_s\rangle\rangle$. Using this fact and the decreasing induction on r one can define the form $\langle\langle\alpha_r\rangle\rangle - \langle\langle\alpha_{r+1}\rangle\rangle + \ldots \pm \langle\langle\alpha_s\rangle\rangle$ as a subform (and a direct summand) of $\langle\langle\alpha_r\rangle\rangle$ orthogonal to $\langle\langle\alpha_{r+1}\rangle\rangle - \ldots \mp \langle\langle\alpha_s\rangle\rangle$. It follows from the definition that the dimension of the obtained form will be exactly $d = 2^{r_1} - 2^{r_2} + \ldots \pm 2^{r_s}$.

Examples:

$d = 2^n$: then *excellent form* is just the *Pfister form*

$d = 5$: the form $\langle 1, -c, ac, bc, -abc\rangle$ is excellent, $a, b, c, \in k^*$.
$r_1 = 3, r_2 = 2, r_3 = 0$, $\alpha_1 = \{a, b, c\}$, $\alpha_2 = \{a, b\}$, $\alpha_3 = 1 = \{\emptyset\}$.

$d = 6$: the form $\langle\langle a\rangle\rangle \cdot \langle -b, -c, bc\rangle$ is excellent. $r_1 = 3, r_2 = 1$,
$\alpha_1 = \{a, b, c\}$, $\alpha_2 = \{a\}$.

We observe that each excellent form produces invariants (which determine it, in turn): numbers $r_1, \dots, r_s$, and pure symbols $\alpha_1 \in \mathrm{K}^M_{r_1}(k)/2, \dots, \alpha_s \in \mathrm{K}^M_{r_s}(k)/2$.

So, as the first approximation, we can expect that each quadratic form produces a series of invariants living in the groups of the type $K_0, K_1, K_2, \dots$, where invariants of type K_0 are *discrete invariants* taking values in the discrete groups (collection of integers), and the invariants of type K_1, K_2, etc. ... are taking values in more and more "continuous groups" (where we count K_2 more continuous than K_1).

Lecture 2

Chow groups and motives

We will be working with the *algebraic varieties*, which we always assume to be *quasiprojective*. Quasiprojective variety is an open subvariety in the *projective variety*. And the latter one is just a closed subvariety of the projective space $\mathbb{P}^N$, that is subvariety given by the set of (homogeneous) equations $f_1, \dots, f_r$. The same quasiprojective variety can be embedded in different projective spaces: $\mathbb{P}^N \supset X \subset \mathbb{P}^M$ (in particular, one can define precisely when such subvarieties are isomorphic).

Algebraic variety can be covered by *affine* open subvarieties. Affine varieties correspond to commutative rings (finitely generated, in our case). This correspondence has the form

$$R - \text{ring} \quad \leftrightarrow \quad \mathrm{Spec}(R),$$

where $\mathrm{Spec}(R)$ is called the *spectrum* of R, and R, in turn is a ring of regular functions on the algebraic variety $\mathrm{Spec}(R)$. The above correspondence is contravariant:

$$\phi : S \to R \quad \leftrightarrow \quad \phi^\vee : \mathrm{Spec}(R) \to \mathrm{Spec}(S).$$

In our situation, affine varieties are just the closed subvarieties of affine space $\mathbb{A}^n = \mathrm{Spec}(k[x_1, \dots, x_n])$ which is just the translation into geometric language of the fact that the respective rings are finitely generated and so are the quotient rings of the polynomial ring: $R = k[x_1 \dots, x_n]/(f_1, \dots, f_r)$. Of course, the same variety can be embedded into many different affine spaces - just choose another set of k-algebra generators $y_1, \dots, y_m$ and present R as $k[y_1 \dots, y_m]/(g_1, \dots, g_s)$.

Algebraic varieties have *points*. Points of the affine variety $\mathrm{Spec}(R)$ are the *prime ideals* $P \subset R$ (that is, such ideals that for any $x, y \in R$, $x \cdot y \in P$ implies that either x, or y belongs to P). The morphism of affine varieties $\phi^\vee : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ acts on points: $P \mapsto \phi^{-1}(P)$. If X is covered by affine open varieties $X = \cup_i U_i$, then

$$(\text{ points of } X) = \coprod_i (\text{ points of } U_i)/(\text{ident.}),$$

where we identify points of $U_i \cap U_j$ in U_i and U_j.

In contrast to topology and usual geometry, the points have different *dimensions*. It is sufficient to consider the case of affine variety.

$$\dim(P) = \max\{d \mid \exists \text{ chain } P = P_0 \subset P_1 \subset \dots \subset P_d \text{ of distinct prime ideals}\}.$$

Points of dimension 0 are exactly the *maximal* ideals in R. If R has no zero divisors then the ideal (0) is prime, and the respective point is called the *generic point*. In such a case the *dimension of a variety* is just the dimension of its generic point.

To each point one can assign the *residue field* $k(x)$. Namely, if P is prime, then the subset $T = R \backslash P$ is multiplicative ($T \cdot T \subset T$), and we can localise: RT^{-1} will be a *local ring*, and PT^{-1} is the only maximal ideal in it. $k(P) := RT^{-1}/PT^{-1}$. The dimension of a point is just the transcendence degree $\mathrm{trdeg}(k(P)/k)$ of its residue field over k. Any regular function r on $\mathrm{Spec}(R)$ (that is, an element of R), can be evaluated at P with value in $k(P)$:

$$R \to RT^{-1} \to RT^{-1}/PT^{-1} = k(P).$$

Notice that all these fields $k(P)$ come with the natural embedding $k \subset k(P)$, so if one considers only the case of closed points over algebraically closed field k, then all the residue fields are identified, and the evaluation takes values in the same field k (as one used to).

Example: $X = \mathrm{Spec}(k[x_1, \dots, x_n]) = \mathbb{A}^n$. Then $\dim(X) = n$, residue field of a generic point is the field of rational functions $k(x_1, \dots, x_n)$, and as a maximal chain of prime ideals one can choose

$$(x_1, \dots, x_n) \supset (x_1, \dots, x_{n-1}) \supset \dots \supset (x_1) \supset (0).$$

Algebraic variety is called irreducible if all of its open affine subvarieties are, and an affine variety $X = \mathrm{Spec}(R)$ is irreducible, if and only if R has no 0-divisors (only "one" generic point).

Examples:

1) $\mathrm{Spec}(k[x, y]/(xy))$ is reducible (consists of the union of x-axis and y-axis on x, y-plane - two components).

2) $\mathrm{Spec}(k[x, y]/(y - x^3))$ is irreducible (consists of just one component).

If (as in the examples above) our variety is a hypersurface in the affine space (given by just one equation), then one simply needs to check if the respective polynomial is decomposable, but if the variety is defined by several equations it could be quite difficult to check the irreducibility.

There is $1 - 1$ correspondence

$$\{\,\text{irred. closed subvar. of}\, X\} \leftrightarrow \{\,\text{points of}\, X\}$$

where each point is a *generic point* of some unique closed irreducible subvariety.

Chow groups

Let X be an algebraic variety, then one can define the *Chow group of d-dimensional cycles on X modulo rational equivalence* as

$$\mathrm{CH}_d(X) := \left(\bigoplus_{V \subset X} \mathbb{Z} \cdot [V]\right) /(\,\text{rational equivalence}\,),$$

where V runs over all closed irreducible subvarieties of X of dimension d (that is, over all points of dimension d of X), $[V]$ is just the formal group generator corresponding to V, and the two combinations are called *rationally equivalent* if there exists a combination $W = \sum_l \nu_l \cdot [W_l]$ of $(d+1)$-dimensional irreducible subvarieties on $X \times \mathbb{P}^1$ such that $W|_{X \times \{0\}} = \sum_i \lambda_i \cdot [V_i]$ and $W|_{X \times \{1\}} = \sum_j \mu_j \cdot [U_j]$ (one can give the precise meaning to the notation $W|_{X \times \{a\}}$).

Have the action of various operations on the Chow groups.

Push-forwards

Let $f: X \to Y$ be a map of algebraic varieties. It is called *projective*, if it can be decomposed as:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Y \times \mathbb{P}^n \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where j is a closed embedding.

Examples:

1) Closed embedding is a projective map.
2) $\mathbb{A}^1 \to \mathrm{Spec}(k)$ is not a projective map.
3) X is projective (a closed subvariety in projective space), then any $f: X \to Y$ is projective.

Roughly speaking, f is projective if all the fibers are projective varieties.

If f is projective we have *push-forward maps*

$$f_*: \mathrm{CH}_d(X) \to \mathrm{CH}_d(Y),$$

where if $V \subset X$ is closed irreducible subvariety of X, and $U \subset Y$ is its image under f, then

$$f_*([V]) := \begin{cases} 0, \text{ if } \dim(U) < \dim(V); \\ \deg(k(V)/k(U)) \cdot [U], \text{ if } \dim(U) = \dim(V). \end{cases}$$

The coefficient $\deg(k(V)/k(U))$ here is just the "number of preimages" of the "sufficiently generic" point of U.

One can prove that in the case of projective map such definition respects the rational equivalence.

Warning: if f is not projective one can try to define f_* by the same formula, but the rational equivalence will not be respected.

Pull-backs

Together with the *dimensional* notations one can use the *codimensional* ones. Namely, $\mathrm{codim}(V \subset X) = \dim(X) - \dim(V)$, and we will denote the same Chow groups in two ways:

$$\mathrm{CH}_d(X) = \mathrm{CH}^{\dim(X)-d}(X).$$

Variety X is called *smooth* if locally it can be defined by $(n - \dim(X))$ equations $f_1, \ldots, f_{n-\dim(X)}$ in some $\mathbb{A}^n$, so that the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ has (maximal possible) rank $(n - \dim(X))$ everywhere on X.

Examples:

1) Projective space is smooth.

2) q-nondegenerate quadratic form, then the respective projective quadric Q is smooth. If q is degenerate, then Q is <u>not</u> smooth.

3) $\mathrm{Spec}(k[x, y]/(y^2 - x^3))$ is <u>not</u> smooth (singularity at $(0, 0)$).

If <u>Y is smooth</u>, one has *pull-back maps*

$$f^* : \mathrm{CH}^c(Y) \to \mathrm{CH}^c(X).$$

For arbitrary f it is not easy to see, how f^* acts on classes of subvarieties, but if f is *smooth morphism* (roughly speaking, all the fibers are smooth varieties)(or even *flat morphism*), then $f^*([U]) = [f^{-1}(U)]$.

For arbitrary varieties X and Y one has the *external product*

$$\mathrm{CH}^a(X) \times \mathrm{CH}^b(Y) \xrightarrow{\times} \mathrm{CH}^{a+b}(X \times Y),$$

given by $[V] \times [U] \mapsto [V \times U]$. If now X is smooth we can combine this product with the pull-back along the diagonal morphism $\Delta : X \to X \times X$ to get a product structure on $\mathrm{CH}^*(X)$.

$$\begin{tikzcd}
\mathrm{CH}^a(X) \times \mathrm{CH}^b(X) \arrow[r, "\times"] \arrow[dr, "\cdot"'] & \mathrm{CH}^{a+b}(X \times X) \arrow[d, "\Delta^*"] \\
& \mathrm{CH}^{a+b}(X).
\end{tikzcd}$$

This gives the structure of the associative commutative ring on $\mathrm{CH}^*(X)$ for smooth variety X.

Category of Chow motives

<u>Category of correspondences</u>

Define $\mathcal{C}(k)$ - the *category of correspondences*:

$Ob(\mathcal{C}(k)) = \{$ smooth proj. var.$over$k$\} \ni [X]$ - typical representative.

$Mor_{\mathcal{C}(k)}([X], [Y]) = \mathrm{CH}_{\dim(X)}(X \times Y)$, where we assume X - connected.

<u>composition:</u>

Let $\varphi \in Mor_{\mathcal{C}(k)}([X],[Y]), \psi \in Mor_{\mathcal{C}(k)}([Y],[Z])$, in other words, $\varphi \in \mathrm{CH}_{\dim(X)}(X \times Y)$, $\psi \in \mathrm{CH}_{\dim(Y)}(Y \times Z)$.

Consider the natural projections

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "\pi_{X,Y}"'] \arrow[d, "\pi_{X,Z}"] \arrow[dr, "\pi_{Y,Z}"] & \\ X \times Y & X \times Z & Y \times Z. \end{tikzcd}$$

Then the composition is defined as:

$$\psi \circ \varphi := ((\pi_{X,Z})_*((\pi_{X,Y})^*(\varphi) \cdot (\pi_{Y,Z})^*(\psi)).$$

It follows from the standard properties of pull-backs and push-forwards, that this operation is associative.

In particular, one gets the associative ring structure on $\mathrm{CH}^{\dim(X)}(X \times X)$. Warning: do not mess it with the product ring structure on $\mathrm{CH}^*(X \times X)$ - our new composition product is almost never commutative, while the product structure is.

Have a natural functor

$$Sm.Proj./k \xrightarrow{C} \mathcal{C}(k)$$

from the category of smooth quasiprojective varieties over k to $\mathcal{C}(k)$, where $X \mapsto [X]$, and $(f : X \to Y) \mapsto [\Gamma_f]$, where $\Gamma_f \subset X \times Y$ is the *graph* of the map f. It is not difficult to check that this is really a functor (respects the composition).

Category of correspondences has a structure of tensor additive category, where $[X] \oplus [Y] := [X \coprod Y]$ (the class of the disjoint union), and $[X] \otimes [Y] := [X \times Y]$.

Now, one can define the *category of effective Chow-motives* over k as the *Karoubian envelope* of $\mathcal{C}(k)$:

$$\mathrm{Chow}^{eff}(k) := Kar(\mathcal{C}(k)),$$

where the Karoubian (=*pseudo-abelian*) envelope of an additive category $\mathcal{C}$ is defined as follows. $p \in \mathrm{End}_{\mathcal{C}}(A)$ is called *projector*, if $p \circ p = p$. The $Kar(\mathcal{C})$ is a category such that

$$Ob(Kar(\mathcal{C})) = \{(A, p), A \in Ob(\mathcal{C}), p \in \mathrm{End}_{\mathcal{C}}(A) \,\text{is a projector}\,\}.$$

$$Mor_{Kar(\mathcal{C})}((A, p), (B, q)) = q \circ Mor_{\mathcal{C}}(A, B) \circ p \subset Mor_{\mathcal{C}}(A, B),$$

and the composition is induced by that in $\mathcal{C}$.

There is natural functor $\mathcal{C}(k) \stackrel{Kar}{\rightarrow} \mathrm{Chow}^{eff}(k)$ sending $[X]$ to the pair $([X], id)$, and the structure of tensor additive category descends from $\mathcal{C}(k)$ to $\mathrm{Chow}^{eff}(k)$. The composition of functors C and Kar gives a *motivic functor* from the category of smooth projective varieties over k to the category of the effective Chow motives.

$$\begin{tikzcd} & \mathcal{C}(k) \arrow[dr, "Kar"] & \\ Sm.Proj./k \arrow[ur, "C"] \arrow[rr, "M"'] & & \mathrm{Chow}^{eff}(k). \end{tikzcd}$$

For the smooth projective variety X we will call its image $M(X)$ - the *motive of* X.

Lecture 3

Motives of quadrics

We have a motivic functor

$$Sm.Proj./k \stackrel{M}{\rightarrow} \mathrm{Chow}^{eff}(k),$$

which provides each smooth projective variety with its invariant *the motive*.

In $\mathrm{Chow}^{eff}(k)$ we get new objects - the direct summands in the motives of smooth projective varieties. In particular, $M(\mathbb{P}^1)$ will be decomposable there. Notice, that $M(\mathbb{P}^1)$ is given by the pair $([\mathbb{P}^1], [\Delta(\mathbb{P}^1)] \in \mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1))$, but in $\mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$ the class $[\Delta(\mathbb{P}^1)]$ is equal to the sum $[pt \times \mathbb{P}^1] + [\mathbb{P}^1 \times pt]$ of two mutually orthogonal projectors (with respect to the composition operation $\circ$), where pt is any k-rational point on $\mathbb{P}^1$. Thus, $M(\mathbb{P}^1) = ([\mathbb{P}^1], [\mathbb{P}^1 \times pt]) \oplus ([\mathbb{P}^1], [pt \times \mathbb{P}^1])$. The first summand here is isomorphic to $M(\mathrm{Spec}(k))$ and will be denoted $\mathbb{Z}(0)[0]$ (or, simply, $\mathbb{Z}$) - the *trivial Tate-motive*, and the second is denoted $\mathbb{Z}(1)[2]$ - the *Tate-motive*. $\mathrm{Chow}^{eff}(k)$ is tensor additive category with $M(X) \oplus M(Y) = M(X \coprod Y)$, and $M(X) \otimes M(Y) = M(X \times Y)$. Can define Tate-motive $\mathbb{Z}(n)[2n]$ as $(\mathbb{Z}(1)[2])^{\otimes n}$. It is given as a direct summand in the motive of $(\mathbb{P}^1)^n$, but will be also a direct summand in the motive $M(X)$ of any smooth projective n-dimensional variety X which has a k-rational point - the respective projector is given by $[pt \times X]$ (in reality, you just need a zero-cycle of degree 1).

Inside $\mathrm{Chow}^{eff}(k)$ you will meet only Tate motives $\mathbb{Z}(n)[m]$ with $m = 2n$. But $\mathrm{Chow}^{eff}(k)$ is naturally a full additive subcategory of the bigger *triangulated category of motives* $DM_{-}^{eff}(k)$, and the latter category already contains Tate-motives $\mathbb{Z}(n)[m]$ with all possible m and n. This is why we will keep both numbers in the notation of Tate-motives, although, in our situation, these numbers are not independent. We get

$$M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2].$$

And, in the same way,

$$M(\mathbb{P}^r) = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \ldots \oplus \mathbb{Z}(r)[2r].$$

with the projectors $[\mathbb{P}^s \times \mathbb{P}^{r-s}]$, for $0 \leqslant s \leqslant r$.

Connection to Chow groups and motivic cohomology

For smooth projective varieties one can naturally identify:

$$\begin{aligned} \mathrm{CH}^n(X) &= \mathrm{Hom}_{\mathrm{Chow}^{eff}(k)}(M(X), \mathbb{Z}(n)[2n]); \\ \mathrm{CH}_n(X) &= \mathrm{Hom}_{\mathrm{Chow}^{eff}(k)}(\mathbb{Z}(n)[2n], M(X)). \end{aligned}$$

and since $\mathrm{Chow}^{eff}(k)$ is a full subcategory of $DM_{-}^{eff}(k)$, the former group can be identified with $\mathrm{Hom}_{DM_{-}^{eff}(k)}(M(X), \mathbb{Z}(n)[2n]) = \mathrm{H}_{\mathcal{M}}^{2n,n}(X, \mathbb{Z})$ - the *motivic cohomology*. Thus we see that

$$\mathrm{CH}^n(X) = \mathrm{H}_{\mathcal{M}}^{2n,n}(X, \mathbb{Z}).$$

Quadrics

The motive of a quadric is the simplest when the quadric is completely split. In this case, it can be decomposed into the direct sum of Tate-motives.

$$M(Q) = \oplus_{i=0}^{[\frac{\dim(Q)}{2}]}(\mathbb{Z}(i)[2i] \oplus \mathbb{Z}(\dim(Q) - i)[2\dim(Q) - 2i])$$

The respective projectors have the form $[l_i \times h^i]$ and $[h^i \times l_i]$, where h^i is a plane section of codimension i on Q, and l_i is a projective subspace of dimension i on Q (which exists since Q is completely split). In particular, one can observe that the motive of odd-dimensional split quadric coincides with the motive of the projective space of the same dimension, although, as algebraic varieties they are not isomorphic (when dimension > 1). This shows that the variety cannot be reconstructed from its motive, in general.

Using the fact that

$$\mathrm{Hom}_{\mathrm{Chow}^{eff}(k)}(\mathbb{Z}(i)[2i], \mathbb{Z}(j)[2j]) = \begin{cases} 0, i \neq j; \\ \mathbf{Z}, i = j. \end{cases}$$

we can compute Chow groups of Q:

$$\mathrm{CH}^i(Q) = \begin{cases} \mathbf{Z}, 0 \leqslant i \leqslant \dim(Q), i \neq \dim(Q)/2; \\ \mathbf{Z} \oplus \mathbf{Z}, i = \dim(Q)/2; \\ 0, \text{ otherwise.} \end{cases}$$

Examples:

1) C - split conic, $M(C) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$;

2) Q - split 2-dimensional quadric, $M(Q) = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(2)[4]$.

What if quadric is not completely split, but just isotropic? Let $q = \mathbb{H} \perp q'$. Then

$$M(Q) = \mathbb{Z} \oplus M(Q')(1)[2] \oplus \mathbb{Z}(\dim(Q))[2\dim(Q)]$$

Applying inductively this fact one gets the case of the split quadric above. Also, this shows that the motive of a quadric can be expressed in terms of the Tate-motives and the motive of the anisotropic part of it.

But what if the quadric is anisotropic, can we still say something about its motive?

Consider the case of a conic C. First of all we observe the following simple fact:

$$C \text{ has a } k\text{- rational point} \quad \Leftrightarrow \quad C \cong \mathbb{P}^1$$

Indeed, the ($\Leftarrow$) conclusion is obvious, since $\mathbb{P}^1$ has plenty of k-rational points. Conversely, let $x \in C$ be some k-rational point. Then C is naturally identified with the $\mathbb{P}^1$ of projective lines on $\mathbb{P}^2$ passing through $x \subset C \subset \mathbb{P}^2$. Thus, if conic is somewhat interesting (do not coincide with the projective line, at least), then it has no rational points.

Suppose that C is arbitrary conic given by some equation $Ax_o^2 + Bx_1^2 + Cx_2^2$. We can divide it by A and get $x_0^2 - ax_1^2 - bx_2^2$ ($a = -B/A, b = -C/A$), so that our form is $\langle 1, -a, -b \rangle$. Then it is a subform of a Pfister form $\langle\langle a, b \rangle\rangle$. By the Main property of Pfister forms, for arbitrary field extension E/k,

$$\langle\langle a, b \rangle\rangle|_E \text{ is isotropic} \Leftrightarrow \langle\langle a, b \rangle\rangle|_E \text{ is completely split.}$$

Hence, this condition is also equivalent to: $\langle 1, -a, -b\rangle|_E$ is isotropic. Really, isotropy of $\langle 1, -a, -b\rangle|_E$ implies isotropy of $\langle\langle a, b\rangle\rangle|_E$ since the former is a subform of the latter. In the other direction, if $\langle\langle a, b\rangle\rangle|_E$ isotropic, then it is completely split, that is, has a totally isotropic subspace of dimension 2, but then such subspace should intersect non-trivially with the 1-codimensional subform $\langle 1, -a, -b\rangle|_E$ to produce isotropic vector for the latter.

Now, we can also remind, that for arbitrary field extension E/k,

$$\langle\langle a, b\rangle\rangle|_E \text{ is completely split } \Leftrightarrow \{a, b\}|_E = 0.$$

This shows that our conic $C_{\{a,b\}}$ and the Pfister quadric $Q_{\{a,b\}}$ are the *norm-varieties* for the pure symbol $\{a, b\} \in \mathrm{K}_2^M(k)/2$. A variety X is called a norm-variety for $\alpha \in \mathrm{K}_n^M(k)/r$ if for arbitrary field extension E/k, $X|_E$ has E-rational point if and only if $\alpha|_E = 0 \in \mathrm{K}_n^M(E)/r$.

Exactly the same considerations show that for arbitrary subform p of $\langle\langle a_1, \ldots, a_n\rangle\rangle$ of dimension $> \frac{1}{2}\dim(\langle\langle a_1, \ldots, a_n\rangle\rangle) = 2^{n-1}$, the respective projective quadric will be a norm-variety for the symbol $\{a_1, \ldots, a_n\} \in \mathrm{K}_n^M(k)/2$. Notice that we have many different varieties corresponding to the same symbol. It is clear that all of them have something in common. And this something appears to be certain direct summand in their motives.

Consider again the case of 2-dimensional 2-fold Pfister quadric $Q_{\{a,b\}}$. Since determinant of $\langle\langle a, b\rangle\rangle$ is 1, the projective lines on $Q_{\{a,b\}}$ split into two families, each of which is naturally identified with $C_{\{a,b\}}$ (each line intersects $C_{\{a,b\}}$ in a unique point - this defines the identification). This simultaneously shows that $Q_{\{a,b\}} = C_{\{a,b\}} \times C_{\{a,b\}}$ (since each point on $Q_{\{a,b\}}$ is determined uniquely by the pair of projective lines on $Q_{\{a,b\}}$ (one from each of the two families) passing through it), and identifies it with $\mathbb{P}_{C_{\{a,b\}}}(\mathcal{V})$ - the projectivisation of certain 2-dimensional vector bundle on $C_{\{a,b\}}$ (since there is a natural projection $Q_{\{a,b\}} \to C_{\{a,b\}}$ given by the lines of one of the families, with the fibers - those lines). It follows from the general theory that the motive of the projective bundle is a direct summand of several copies of the motive of the base with various Tate-twists (for $U \in Ob(\mathrm{Chow}^{eff}(k))$ we call $U(n)[2n] := U \otimes \mathbb{Z}(n)[2n]$ - the *Tate-twist* of U). In our situation, we get:

$$M(Q_{\{a,b\}}) = M(C_{\{a,b\}}) \oplus M(C_{\{a,b\}})(1)[2].$$

This is the first example of the following general result obtained by M. Rost:

Theorem 0.5 (M. Rost) *Let $\alpha \in \mathrm{K}_n^M(k)/2$ be the pure symbol, and Q_α be the respective Pfister form. Then there exists such motive $M_\alpha \in Ob(\mathrm{Chow}^{eff}(k))$*

that

$$M(Q_\alpha) = \oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)[2i],$$

(then it is easy to see that $M_\alpha|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(2^{n-1}-1)[2^n-2]$*), and* M_α *splits into the sum of Tate-motives if and only if* $\alpha = 0$.

The motive M_α is called the *Rost motive.*

Examples:

1) $n = 1$, then $M_{\{a\}} = M(\mathrm{Spec}(k\sqrt{a}))$;

2) $n = 2$, then $M_{\{a,b\}} = M(C_{\{a,b\}})$.

3) For $n > 3$, M_α is no longer represented by the motive of any algebraic variety, but only by a direct summand in such.

M. Rost also had shown that M_α is a direct summand in the motives of any subquadrics of Q_α of codimension $< 2^{n-1}$ (such subquadrics are called *Pfister neighbours*). Let q_α be n-fold Pfister form, $p \subset q_\alpha$ a subform of dimension $2^{n-1}+m$, $m > 0$, and $p^\perp$ be the *complimentary form* ($q_\alpha = p \perp p^\perp$). Then

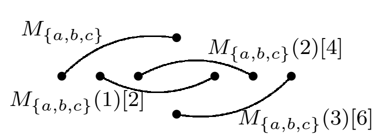
$$M(P) = \oplus_{i=0}^{m-1} M_\alpha(i)[2i] \oplus M(P^\perp)(m)[2m].$$

And the appearance of M_α in this decomposition explains why all such quadrics are the norm-varieties for the pure symbol α. Namely, the existence of a rational point on P is equivalent to $M(P)$ containing Tate-motive $\mathbb{Z}$ as a direct summand (follows from the Theorem of Springer), and is further equivalent to M_α containing such a summand - equivalent to M_α being split, which happens if and only if $\alpha = 0$.

Applying the above statement inductively, one gets that the motive of an excellent quadric is a sum of Rost-motives (of different foldness).

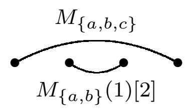
Examples:

1) The motive of 3-fold Pfister form $Q_{\{a,b,c\}}$ can be visualised as

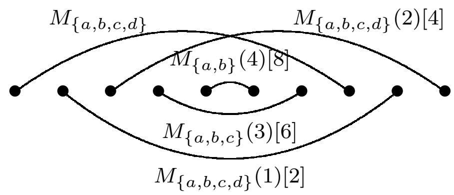


where each $\bullet$ represents a Tate-motive over $\overline{k}$, ranging from $\mathbb{Z}$ on the left to $\mathbb{Z}(6)[12]$ on the right, and each pair of connected $\bullet$'s represents the copy of the Rost-motive $M_{\{a,b,c\}}(i)[2i]$.

2) Let q be 5-dimensional excellent form $\langle 1, -c, ac, bc, -abc\rangle$, then $M(Q)$ can be visualised as



3) Let q be 11-dimensional excellent form $(\langle\langle a,b,c,d\rangle\rangle\perp-\langle\langle a,b,c\rangle\rangle\perp\langle\langle a,b\rangle\rangle\perp -\langle 1\rangle)_{an.}$ (we assume a, b, c, d algebraically independent). Then $M(Q)$ looks as



Hypothetically, the Rost-motives are the only possible *binary direct summands* (that is, motives, which split into the direct sum of exactly 2 Tate-motives over $\overline{k}$) in the motives of quadrics, and the excellent forms are the only forms whose motives split into binary direct summands.

Motivic decomposition type

Definition 0.6 *For the quadric Q let us denote as $\Lambda(Q)$ the set of Tate-motives in the decomposition of its motive over $\overline{k}$:*

$$M(Q|_{\overline{k}}) = \oplus_{\lambda\in\Lambda(Q)}\mathbb{Z}(i_\lambda)[2i_\lambda].$$

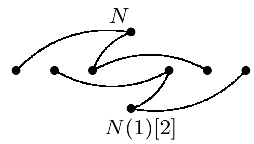
Then for any direct summand N of $M(Q)$ we can identify the set $\Lambda(N)$ of Tate-motives in the decomposition of $N|_{\overline{k}}$ with the subset of $\Lambda(Q)$. We say that $\lambda\in\Lambda(Q)$ and $\mu\in\Lambda(Q)$ are connected, if for any direct summand N of $M(Q)$, $\lambda\in\Lambda(N)\Leftrightarrow\mu\in\Lambda(N)$. The presentation of $\Lambda(Q)$ as the disjoint union of its connected components is called motivic decomposition type *of Q - $MDT(Q)$.*

The *motivic decomposition type* can be visualised as a picture of the same sort as above.

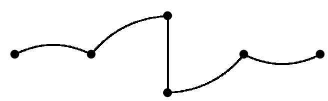
Examples:

1) Let $q = \langle\langle a\rangle\rangle\cdot\langle b,c,d,e\rangle$, where a, b, c, d are algebraically independent. Then $M(Q)$ splits into the sum of two (isomorphic up to Tate-shift)

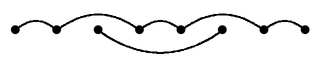
indecomposable direct summands, and $MDT(Q)$ looks as



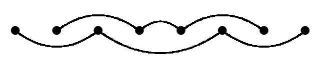
2) Let q be *Albert form* $\langle a, b, -ab, -c, -d, cd\rangle$. Then $M(Q)$ is indecomposable, and $MDT(Q)$ consists of one connected component:



3) Let q be 9-dimensional form $(\langle\langle a, b, c\rangle\rangle \perp -\langle 1, -d, -e\rangle)_{an}$, where a, b, c, d, e are algebraically independent. Then $MDT(Q)$ looks as:



4) Let q be 9-dimensional form $\langle\langle a\rangle\rangle \cdot \langle b, c, d, e\rangle \perp \langle 1\rangle$, where a, b, c, d, e are algebraically independent. Then $MDT(Q)$ looks as:



Splitting pattern

Another discrete invariant of quadrics is the *splitting pattern* invariant. Introduced by M. Knebusch, U. Rehmann and J. Hurrelbrink, it measures what are possible *Witt-indices* $i_W(q|_E)$ of our form over all possible field extensions E/k. One gets the increasing sequence of natural numbers $j_0 < j_1 < j_2 < \ldots < j_h$ - the possible values of $i_W(q|_E)$. The numbers $i_l := j_l - j_{l-1}, l \geqslant 1$ are called the *higher Witt indices*. Assuming q-anisotropic ($j_0 = 0$), the sequence $(i_1, i_2, \ldots, i_h)$ is called the *splitting pattern* $SP(Q)$. The number h is called the *height* of Q.

Examples:

1) For the n-fold Pfister form q_α, $SP(Q_\alpha) = (2^{n-1})$, and the height is 1, since the Pfister form becomes completely split as soon as it is isotropic. The Pfister quadrics and the subquadrics of codimension 1 in them are the only examples of (anisotropic) quadrics of height 1.

2) For Albert form $q = \langle a, b, -ab, -c, -d, cd\rangle$, we have $SP(Q) = (1, 2)$, and $h(Q) = 2$.

3) For the generic form $q = \langle b_1, \dots, b_m\rangle$, where $b_1, \dots, b_m$ are algebraically independent, $SP(Q) = (1, 1, \dots, 1)$, and $h(Q) = [m/2]$.

4) For the form $q = \langle\langle a_1, \dots, a_n\rangle\rangle \cdot \langle b_1, \dots, b_{2r}\rangle$, where $a_1, \dots, a_n, b_1, \dots, b_{2r}$ are algebraically independent, $SP(Q) = (2^n, 2^n, \dots, 2^n)$, and $h(Q) = r$.

5) An (anisotropic) excellent form q of dimension 19 has the splitting pattern $(3, 5, 1)$ and height 3.

It is an important problem in the theory of quadratic forms to find all the possible values of the invariants $MDT(Q)$ and $SP(Q)$. Among the partial results I should mention the Theorem of N. Karpenko, which claims that $(i_1(q)-1)$ should always be a remainder of the division of $(\dim(q)-1)$ by some power of 2. Although, we understand MDT and SP to some extent, there is no even hypothetical description of their possible values. Nevertheless, the interaction between the splitting pattern and motivic decomposition type invariants provides a lot of information about both of them. This suggests that one should try to embed them as faces into some larger invariant, where one can expect to have more structure. In the next lecture we will introduce such big invariant of geometric origin, called *Generic discrete invariant of* Q.

Lecture 4

Generic discrete and elementary discrete invariants of quadrics

On the previous lecture the two discrete invariants of quadrics were introduced: the *motivic decomposition type* and the *splitting pattern*. We will show that both these invariants live inside some big discrete invariant of geometric origin as (rather small) faces. The idea here is, instead of studying the faces, to study the whole invariant, since it should posses more structure. Let us start with $MDT(Q)$. This invariant measures what are possible decompositions of $M(Q)$, that is, what kind of projectors we have in $\mathrm{End}_{\mathrm{Chow}^{eff}(k)}(M(Q))$.

Rost Nilpotence Theorem

The following result of M. Rost is central here:

Theorem 0.7 (RNT)

$$\mathrm{Ker}(\mathrm{End}_{\mathrm{Chow}^{eff}(k)}(M(Q)) \stackrel{\overline{ac}}{\to} \mathrm{End}_{\mathrm{Chow}^{eff}(\overline{k})}(M(Q|_{\overline{k}}))$$

consists of nilpotents.

This implies that any projector in the image of $\overline{ac}$ can be lifted to a projector in $\mathrm{End}_{\mathrm{Chow}^{eff}(k)}(M(Q))$, and two such liftings produce direct summands which are isomorphic as objects of $\mathrm{Chow}^{eff}(k)$. So, to know the decomposition of $M(Q)$ it is sufficient to know the

$$image(\overline{ac}) = image(\mathrm{CH}^{\dim(Q)}(Q \times Q) \to \mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}})).$$

Consider for simplicity the case $\dim(Q)$-odd (the other one can be done similarly). Then $2 \cdot \mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}}) \subset image(\overline{ac})$, since $\mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}})$ is additively generated by $[l_i \times h^i]$, and $2 \cdot l_i = h^{\dim(Q)-i}$, which implies that $[h^{\dim(Q)-i} \times l_i] \in image(\overline{ac})$. Thus, after all, we need to know only the

$$image(\mathrm{CH}^{\dim(Q)}(Q \times Q)/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^{\dim(Q)}(Q \times Q|_{\overline{k}})/2).$$

Example: Let $\dim(Q)$ is odd. Then $M(Q)$ is indecomposable if and only if the image above consists of just $\mathbf{Z}/2 \cdot [\Delta_Q]$.

Aside: RNT shows that $M(Q)$ does not contain *phantom* direct summands. That is, if N is a direct summand, and $N|_{\overline{k}} = 0$, then $N = 0$.

RNT is generalised to the case of arbitrary projective homogeneous variety by V. Chernousov, A. Merkurjev and S. Gille. So, the motives of these varieties also have no phantom direct summands.

Hypothetically, NT should hold for arbitrary smooth projective variety, and so there should be no phantom objects in $\mathrm{Chow}^{eff}(k)$ at all. But this is a very strong and complicated Conjecture (related to the Conjecture of S. Bloch). Notice, that in $DM_{-}^{eff}(k)$ there is plenty of phantom objects, and many of these were successfully used (most notably, by V. Voevodsky), but they are *infinite dimensional* and do not live in $\mathrm{Chow}^{eff}(k)$.

Definition 0.8 *Consider the following invariant of quadrics:*

$$Q \mapsto image(\mathrm{CH}^*(Q^{\times N})/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^*(Q^{\times N}|_{\overline{k}})/2), \text{ for all } N.$$

We call it Generic discrete invariant of quadrics (in non-compact form).

This invariant clearly contains $MDT(Q)$. The disadvantage here is that one has to consider infinitely many objects. But the invariant can be "compactified", and the above problem disappears.

To each smooth projective quadric Q one can assign the respective *quadratic Grassmannians*:

$$Q \mapsto G(i,Q) \ - \ \text{Grassmannian of } i - \text{dim. planes on } Q.$$

This is smooth projective (homogeneous) variety, and E-rational points of $G(i,Q)$ are i-dimensional planes $l_i \subset Q|_E$.

We get varieties:

$$Q = G(0,Q),\ G(1,Q),\dots,G(d,Q),\ \text{where}\, d = \left[\frac{\dim(Q)}{2}\right].$$

Examples:

1) $\dim(q) = 4$, $q = \langle a,b,c,d\rangle$. Then $G(1,Q) = C_{\{-ab,-ac\}} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k\sqrt{abcd})$ - the conic over the quadratic extension. So, $G(1,Q)|_{\overline{k}} = \mathbb{P}^1 \coprod \mathbb{P}^1$.

2) $\dim(q) = 5$, $q = \langle a,b,c,d,e\rangle$. Consider the auxiliary form $p = q \perp \langle -det(q)\rangle = \langle a,b,c,d,e,-abcde\rangle$. Then $\exists\lambda$ (for example $= abc$), such that $\lambda \cdot p = \langle A,B,-AB,-C,-D,CD\rangle$ is an Albert form, corresponding to the bi-quaternion algebra $Al = Quat(\{A,B\},k)\otimes_k Quat(\{C,D\},k)$. Then $G(1,Q) = SB(Al)$ is a Severi-Brauer variety for the algebra Al. In particular, $G(1,Q)|_{\overline{k}} = \mathbb{P}^3$.

3) Let q_α be the 3-fold Pfister form $\langle\langle a,b,c\rangle\rangle$. Then $G(3,Q_\alpha) = Q_\alpha \coprod Q_\alpha$.

It appears that $M(Q^{\times N})$ can be decomposed into the direct sum of the motives of $G(i,Q)$ with various Tate-shifts.

Example:

$$\begin{aligned} M(Q\times Q) \ &= \ M(Q) \oplus (M(G(1,Q)) \oplus M(G(1,Q))(1)[2]) \\ &\oplus M(Q)(\dim(Q))[2\dim(Q)]. \end{aligned}$$

Consequently, to know

$$image(\mathrm{CH}^*(Q^{\times N})/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^*(Q^{\times N}|_{\overline{k}})/2),\ \text{for all}\, N$$

is the same as to know

$$image(\mathrm{CH}^*(G(i,Q))/2 \stackrel{\overline{ac}}{\to} \mathrm{CH}^*(G(i,Q)|_{\overline{k}})/2),\ \text{for}\, 0 \leqslant i \leqslant d = \left[\frac{\dim(Q)}{2}\right]$$

Definition 0.9 *This invariant is called Generic discrete invariant (in compact form) $GDI(Q)$.*

It contains not just $MDT(Q)$, but the $SP(Q)$ as well. Recall, that the Splitting Pattern of Q measures what are possible *Witt-indices* of $q|_E$ for all possible field extensions E/k. It follows from the *Specialisation theory* of M. Knebusch, that it is sufficient to consider only the fields $E = k(G(i,Q)), 0 \leqslant i \leqslant d$ - the generic points of quadratic Grassmannians. In the end, one needs only to know, for which i there is a rational map $G(i,Q) \dashrightarrow G(i+1,Q)$, or, which is the same, the rational section of the projection $F(i,i+1,Q) \to G(i,Q)$ (from the variety of flags $(l_i \subset l_{i+1})$ to the Grassmannian of i-planes on Q). Due to the Theorem of Springer (claiming that Q is isotropic $\Leftrightarrow$ it has a zero-cycle of degree 1) this can be reduced to the existence of cycles of certain type in $\mathrm{CH}^*(F(i,i+1;Q))/2$. But $F(i,i+1;Q)$ is a projective bundle over $G(i+1,Q)$ and, consequently, $M(F(i,i+1;Q))$ is a direct sum of $M(G(i+1,Q))$ with various Tate-shifts. Thus, $GDI(Q)$ contains $SP(Q)$.

Varieties $G(i,Q)$ are *geometrically cellular*, that is, over $\overline{k}$ they can be "cut" into pieces isomorphic to affine spaces $\mathbb{A}^{r_j}$ - *Schubert cells* (to define such a cell, fix a complete flag $\pi_0 \subset \pi_1 \subset \ldots \subset \pi_d$, and natural numbers $n_0, \ldots, n_d$, then the $Cell(n_0, \ldots, n_d)$ is given by the locus of those i-planes l_i that $\dim(l_i \cap \pi_j) = n_j$). Thus, $M(G(i,Q)|_{\overline{k}})$ is (canonically!) a sum of Tate-motives, and $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})$ is a free abelian group with the canonical basis corresponding to Schubert cells

$$Cell \mapsto [\overline{Cell}] \in \mathrm{CH}^*(G(i,Q)|_{\overline{k}}).$$

The Schubert cells are parametrised by some sort of *Young diagrams*, and this way the ring $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})/2$ appears as quite combinatorial object. $GDI(Q,i)$ is the subring of $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})/2$ consisting of elements defined over k. But the ring $\mathrm{CH}^*(G(i,Q)|_{\overline{k}})/2$ is still rather large. For example, for $i = d$ it has the rank $= 2^{d+1}$. Need something handier. For this purpose there is $EDI(Q)$ - *Elementary discrete invariant of* Q. It does not determine the whole $image(\overline{ac})$, but just checks if some particular good classes are in the image, or not. These classes are *elementary classes*. To define them , start with the Grassmannian of 0-dimensional planes $G(0,Q)$, that is, with the quadric Q itself. Elementary classes on Q are just the classes $l_0, l_1, \ldots, l_d$ in $\mathrm{CH}^*(Q|_{\overline{k}})/2$ - these are the only interesting classes there (their k-rationality measures only the Witt-index of q). Now, the elementary classes on other Grassmannians can be produced from that on Q. Namely, we have natural projections:

$$Q \xleftarrow{\alpha_i} F(0,i;Q) \xrightarrow{\beta_i} G(i,Q)$$

Definition 0.10 *Define the elementary classes*

$$y_{i,j} := (\beta_i)_*(\alpha_i)^*(l_j) \in \mathrm{CH}^{\dim(Q)-i-j}(G(i,Q)|_{\overline{k}})/2.$$

$EDI(Q)$ measures which of $y_{i,j}$ are defined over k.

Our elementary classes are numbered by $0 \leqslant i, j \leqslant d$, so $EDI(Q)$ can be visualised as $d \times d$ square, where integral node is marked iff the respective class $y_{i,j}$ is defined over k.

$$\begin{array}{c|cccccc}
 & \bullet & \bullet & \circ & \cdots & \circ & \bullet \\
 & \circ & \bullet & \circ & \cdots & \circ & \circ \\
 & \circ & \circ & \circ & \cdots & \circ & \circ \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\uparrow & \circ & \circ & \circ & \cdots & \circ & \circ \\
i & \circ & \circ & \circ & \cdots & \circ & \circ \\
\hline
 & \xrightarrow{\quad j \quad} & & & & &
\end{array}$$

Here each row corresponds to a particular Grassmannian, and codimension decreases up and right. SW corner is marked $\Leftrightarrow$ Q is isotropic; SE corner is marked $\Leftrightarrow$ it is completely split.

Examples:

1) q-generic $(\langle a_1, \ldots, a_n\rangle/k = F(a_1, \ldots, a_n))$. Then $EDI(Q)$ is empty.

2) q-completely split $\Rightarrow$ everything is marked.

3) q_α is (anisotropic) n-fold Pfister form. Then the marked points will be exactly those which live strictly above the main (NW-SE) diagonal. In the case of $n = 3$ we get:

$$\begin{array}{cccc}
\circ & \bullet & \bullet & \bullet \\
\circ & \circ & \bullet & \bullet \\
\circ & \circ & \circ & \bullet \\
\circ & \circ & \circ & \circ
\end{array}$$

4) The $EDI(Q)$ for the 10-dimensional excellent form looks as:

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \circ \\
\bullet & \bullet & \circ & \bullet & \circ \\
\bullet & \bullet & \circ & \circ & \circ \\
\circ & \bullet & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ
\end{array}$$

5) Let $q = \langle a, b, -ab, -c, -d, cd\rangle$ be an anisotropic Albert form. Then $EDI(Q)$ is

$$\begin{array}{ccc}
\circ & \circ & \bullet \\
\circ & \circ & \circ \\
\circ & \circ & \circ
\end{array}$$

The serious constraint on such marking is provided by the following:

Rule: $\begin{array}{ccc} \bullet & & \bullet \\ & \nwarrow & \uparrow \\ & & \bullet \end{array}$

Usually, one cannot reconstruct GDI from EDI, but for $i = d$ these two invariants carry the same information due to the following result:

Theorem 0.11 *$GDI(Q, d)$ is generated as a ring by elementary classes contained in it.*

So, instead of studying the subrings of the ring of rank 2^{d+1} it is sufficient to study the subset of the set of $(d + 1)$ elements $(0, 1, \ldots, d)$, where $j \leftrightarrow y_{d,j}$. Moreover, the action of the Steenrod operations on $GDI(Q, d)$ preserves the elementary classes, and so, provides the action on $EDI(Q, d)$. Hypothetically, the restrictions coming from this action (j-defined, $\binom{j}{r}$-odd $\Rightarrow$ $(j + r)$-defined) are the only restrictions on the possible subsets.

For other Grassmannians nothing of this sort is true. In particular, the elementary classes do not determine GDI, and the rigidity structure on GDI should involve all the Grassmannians simultaneously (in the contrast to the last Grassmannian being "self-sufficient").

It is an interesting task to translate EDI into the classical quadratic form theory language. Here the dots living below the auxiliary (SW-NE) diagonal are better understood. For such classes ($i \leqslant j+1$) the k-rationality can be hypothetically expressed in terms of *dimensions of B.Kahn*. This important discrete invariant of quadrics is defined as follows:

Definition 0.12

$$\dim_{I^n}(q) = \min(\dim(p)|\, q \perp -p \in I^n).$$

This invariant measures how far is our form from the given power of the fundamental ideal of even-dimensional forms.

Conjecture 0.13 *For $i \leqslant j+1$, the following conditions are equivalent:*

$$y_{i,j} \text{ is } k-\text{rational} \Leftrightarrow \dim_{I^r}(q) \leqslant c,$$

where $c = \dim(Q) - 2j$, and $r = [log_2(\dim(Q) - i - j + 1)] + 2$.

Notice, that the dimensions of B. Kahn one encounters here are all in the *stable range* $< 2^{n-1}$ (for such dimensions the closest point in I^n (and the form p above) is unique - follows from the Arason-Pfister Hauptsatz, claiming that the dimensions of anisotropic forms in I^n are either 0, or $\geqslant 2^n$). It is expected that *unstable dimensions of B. Kahn* should appear when one considers invariant similar to GDI, but with $\mathrm{CH}^*/2$ substituted by the ring of Algebraic Cobordism Ω^* (see the next lecture).

The other half of $EDI(Q)$ $(i > j+1)$ is substantially less understood, and in the known examples the description here involves invariants similar to the Kahn's dimensions, but of more complicated nature.

Lecture 5

Algebraic Cobordism. Landweber-Novikov and Steenrod operations. Symmetric operations.

Let k be a field of characteristic 0, and $(Sm.Q.-P.)/k$ be the category of smooth quasi-projective varieties over k.

The *generalised oriented cohomology theory* is a contra-variant functor

$$(Sm.Q.-P.)/k \xrightarrow{A^*} \{\mathbf{Z}-\text{graded rings}\}$$

$$X \mapsto A^*(X)$$

$$(f: X \to Y) \mapsto (f^*: A^*(Y) \to A^*(X))$$

together with the *push-forward morphisms* $f_*: A^*(X) \to A^{*-d}(Y)$ for projective equidimensional maps $f: X \to Y$ of relative dimension d.

All these data should satisfy certain compatibility axioms.

Generalised oriented cohomology theory possesses Chern classes.

Chern classes

Let $\mathcal{L}/X$ be line bundle. Consider the zero section $j : X \hookrightarrow \mathcal{L}$. Then one can assign to $\mathcal{L}$ its first *Chern class* $c_1(\mathcal{L}) := j^* j_*(1_X^A) \in A^1(X)$.

Now, if $\mathcal{U}$ is some vector bundle on X, by the *projective bundle axiom* of the generalised oriented cohomology theory,

$$A^*(\mathbb{P}_X(\mathcal{U}^\vee)) = \oplus_{i=0}^{\dim(\mathcal{U})-1} \rho^i \cdot A^*(X),$$

where $\rho := c_1(\mathcal{O}(1))$, and $\mathcal{U}^\vee$ is the vector bundle dual to $\mathcal{U}$. In particular, there is the unique relation:

$$\rho^{\dim(\mathcal{U})} - \lambda_1 \cdot \rho^{\dim(\mathcal{U})-1} + \lambda_2 \cdot \rho^{\dim(\mathcal{U})-2} - \ldots (-1)^{\dim(\mathcal{U})} \lambda_{\dim(\mathcal{U})} = 0,$$

for certain $\lambda_i \in A^i(X)$. These coefficients are called the *Chern classes* of the bundle $\mathcal{U}$: $c_i(\mathcal{U}) := \lambda_i$ (we assume $c_0 = \lambda_0 = 1$). Denote as $c_\bullet(\mathcal{U})$ the *total Chern class* $\sum_{i\geqslant 0} c_i(\mathcal{U})$. These classes satisfy the *Cartan formula*: if $0 \to \mathcal{U}_1 \to \mathcal{U}_2 \to \mathcal{U}_3 \to 0$ is a short exact sequence, then

$$c_\bullet(\mathcal{U}_1) \cdot c_\bullet(\mathcal{U}_3) = c_\bullet(\mathcal{U}_2).$$

Cartan formula permits to define Chern classes on the formal differences $\mathcal{V} - \mathcal{U}$ of vector bundles, that is, on K^0.

Examples (of theories):

1) CH^* - the *Chow groups*.

2) $K^0[\beta, \beta^{-1}]$ - the algebraic K^0 (it is convenient to add the formal invertible parameter β to it).

Among such theories there is the universal one Ω^* called *Algebraic Cobordism*. This theory was constructed by M. Levine and F. Morel (further simplified by M. Levine and R. Pandharipande).

$\Omega^*(X)$ is additively generated by the classes $[v : V \to X]$, where V is smooth and v is projective. One imposes certain relations:

1) *Elementary cobordism relations*

The classes $[v_0 : V_0 \to X]$ and $[v_1 : V_1 \to X]$ are *elementary cobordant*, if there exists projective map $w : W \to X \times \mathbb{P}^1$ from a smooth variety W, which is transversal to $X \times \{0\} \hookrightarrow X \times \mathbb{P}^1$ and $X \times \{1\} \hookrightarrow X \times \mathbb{P}^1$ and $w|_{w^{-1}(X\times\{0\})} = v_0$, $w|_{w^{-1}(X\times\{1\})} = v_1$.

We recall that the morphisms f, g from the Cartesian square

$$\begin{array}{ccc} B \times_A C & \xrightarrow{g'} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & A \end{array}$$

with A, B, C - smooth are called transversal, if the natural map of tangent bundles $(f')^*T_B \oplus (g')^*T_C \rightarrow (f \circ g')^*T_A$ is surjective. Then $B \times_A C$ is smooth, and the sequence

$$0 \rightarrow T_{B \times_A C} \rightarrow (f')^*T_B \oplus (g')^*T_C \rightarrow (f \circ g')^*T_A \rightarrow 0$$

is exact. The *transversal Cartesian squares* behave especially well with respect to the pull-back and push-forward morphisms, and they are used in the definition of the generalised oriented cohomology theory.

In topology these would be all the relations, but in algebraic geometry one has to impose more.

2) *Double point relations* (following M. Levine - R. Pandharipande) Let $[w : W \rightarrow X \times \mathbb{P}^1]$ be such projective map that w is transversal to $X \times \{0\} \rightarrow X \times \mathbb{P}^1$, where $w|_{w^{-1}(X \times \{0\})} = v_0 : V_0 \rightarrow X$, and $w^{-1}(X \times \{1\})$ consists of two smooth components $V_{1,a}$ and $V_{1,b}$ intersecting transversally on W at $U = V_{1,a} \cap V_{1,b}$. Let $\mathcal{N} = \mathcal{N}_{U \subset V_{1,a}}$ be the normal bundle (it is easy to see that then $\mathcal{N}_{U \subset V_{1,b}} = \mathcal{N}^{-1}$). Then we impose the relation:

$$[v_0 : V_0 \rightarrow X] = [v_{1,a} : V_{1,a} \rightarrow X] + [v_{1,b} : V_{1,b} \rightarrow X] - [\mathbb{P}_U(\mathcal{N} \oplus \mathcal{O}) \rightarrow X].$$

Notice, that this relation is symmetric with respect to $a \leftrightarrow b$, since $\mathbb{P}_U(\mathcal{N} \oplus \mathcal{O})$ is isomorphic to $\mathbb{P}_U(\mathcal{O} \oplus \mathcal{N}^{-1})$.

One generates all the relations in Ω^* by applying the push-forward operation f_* with respect to all proper morphisms $f : X \rightarrow Y$ to the two types of relations above, where $f_*([v : V \rightarrow X]) := [f \circ v : V \rightarrow Y]$. As was mentioned, the resulting theory Ω^* is universal oriented generalised cohomology theory. The universality follows from the fact that oriented theories have push-forwards: the canonical map

$$\Omega^*(X) \rightarrow A^*(X)$$

is given by

$$[v : V \rightarrow X] \mapsto (v_A)_*(1_V^A) \in A^{codim(V \subset X)}(X).$$

Remark: It should be mentioned, that it is quite nontrivial to define the pull-back operations f^* on Ω^*. One can find details in the book of M. Levine and F. Morel.

Properties of Ω^*

(1) $\Omega^*(\mathrm{Spec}(k)) = MU^{2*}(pt) = \mathbb{L}$, where MU is the $\mathbb{C}$-oriented cobordism in topology, and $\mathbb{L}$ is the *Lazard ring* - the coefficient ring of the *universal formal group law*. In particular, we see that the result does not depend on k (it does not matter, if k is algebraically closed, or not).

Formal group laws

(commutative, 1-dimensional) formal group law is given by the following data: $(R, F(x,y))$, where R is a coefficient ring (associative, commutative, unital), and $F(x,y) \in R[[x,y]]$ is a power series, satisfying:

(i) $F(x,0) = x,\ F(0,y) = y$;

(ii) $F(x,y) = F(y,x)$ - commutativity;

(iii) $F(F(x,y),z) = F(x,F(y,z))$ - associativity.

From conditions (i) and (ii) it follows that $F(x,y) = x + y + \sum_{i,j\geqslant 1} a_{i,j}x^iy^j$, with $a_{i,j} = a_{j,i}$.

Examples:

1) *Additive group law*: $F_a(x,y) = x + y$ with R-any ring;

2) *multiplicative group law*: $F_m(x,y) = x + y - \beta \cdot xy$, where $\beta \in R$ is invertible.

Among the group laws there is the universal one $(R_U, F_U(x,y))$ such that there is $1-1$ correspondence

$$\{\,\text{f.g.laws}\,(R, F(x,y))\} \leftrightarrow \{\,\text{ring homomorphisms } R_U \xrightarrow{f_F} R\},$$

where $F(x,y) = f_F(F_U(x,y))$. Clearly, it is sufficient to take

$$R_U := \mathbf{Z}[a_{i,j},\ i,j \geqslant 1]/(assoc., comm.)$$

with the $F_U = x+y+\sum_{i,j\geqslant 1} a_{i,j}x^iy^j$. The coefficient ring R_U of the universal formal group law is called the *Lazard ring* $\mathbb{L}$. The following important result is due to Lazard:

Theorem 0.14

$$\mathbb{L} = \mathbf{Z}[z_1, z_2, \ldots], \ \textit{with } deg(z_l) = l, \ \textit{where } deg(a_{i,j}) = i + j - 1.$$

Projective bundle axiom implies that $A^*(\mathbb{P}^\infty) = A^*[[t]]$, where $A^* := A^*(\mathrm{Spec}(k))$, and $t = c_1(\mathcal{O}(1))$. Consider the *Segre embedding*

$$\mathbb{P}^\infty \times \mathbb{P}^\infty \stackrel{Segre}{\longrightarrow} \mathbb{P}^\infty.$$

It induces the pull-back homomorphism

$$A^*[[x, y]] \stackrel{Segre^*}{\longleftarrow} A^*[[t]].$$

It is easy to check that the pair $(A^*, F_A(x, y))$, where $F_A(x, y) := (Segre)^*(t)$ will be a formal group law. Thus to each generalised cohomology theory one can assign the formal group law

$$A^*(X) \mapsto (A^*, F_A(x, y)).$$

Examples:

1) $\mathrm{CH}^* \mapsto (\mathbf{Z}, F_a(x, y))$;

2) $K^0[\beta, \beta^{-1}] \mapsto (\mathbf{Z}[\beta, \beta^{-1}], F_m(x, y))$.

Due to the result of M. Levine and F. Morel, the theories CH^* and $K^0[\beta, \beta^{-1}]$ are the universal ones among the *additive* and *multiplicative* theories, respectively.

The formal group law assigned to the theory A^* describes how the 1-st Chern class behaves with respect to $\otimes$ operation for linear bundles:

$$c_1^A(\mathcal{L} \otimes \mathcal{M}) = F_A(c_1^A(\mathcal{L}), c_1^A(\mathcal{M})).$$

It appears that the formal group law assigned to the Algebraic Cobordism theory Ω^* will be the universal one. That is, $F_\Omega(x, y) = F_U(x, y)$, and $\Omega^*(\mathrm{Spec}(k)) = \mathbb{L}$. In particular, since $\Omega^*(\mathrm{Spec}(k))$ is additively generated by the classes of smooth projective varieties over k, the universal constants $a_{i,j}$ can be interpreted as $\mathbf{Z}$-linear combinations of such classes.

Examples:

1) $a_{1,1} = -[\mathbb{P}^1]$;

2) $a_{2,1} = [\mathbb{P}^1 \times \mathbb{P}^1] - [\mathbb{P}^2]$.

In general, $\dim(a_{i,j}) = i + j - 1$.

(2) We have canonical map of theories $pr : \Omega^* \to \mathrm{CH}^*$ given by

$$[v : V \to X] \mapsto v_*(1_V) \in \mathrm{CH}_{\dim(V)}(X).$$

There is the following important result of M. Levine and F. Morel:

Theorem 0.15

$$\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}^{<0} \cdot \Omega^*(X).$$

Thus, CH^* can be computed out of Ω^*.

Remark: The topological counterpart of this statement, as well as the one with the motivic cohomology in place of the Chow groups are false.

Landweber-Novikov operations

Let $R(\sigma_1, \sigma_2, \ldots) \in \mathbb{L}[\sigma_1, \sigma_2, \ldots]$ be some polynomial, where we assign grading: $\deg(\sigma_i) = i$. Then one can define *Landweber-Novikov operation*

$$S^R_{L-N} : \Omega^*(X) \to \Omega^{*+\deg(R)}(X)$$

by the rule: $S^R_{L-N}([v : V \to X]) := v_*(R(c_1, c_2, \ldots))$, where $c_i = c_i(\mathcal{N}_v) \in \Omega^i(V)$, and $\mathcal{N}_v := -T_V + v^*T_X$ - the *virtual normal bundle*.

There is another parametrisation of Landweber-Novikov operations - the one using *partitions*. Partition is the non-ordered set of natural numbers $\overline{a} = (a_1, a_2, \ldots, a_m)$ with $|\overline{a}| = \sum_i a_i$. To each partition $\overline{a}$ one can assign the minimal symmetric polynomial, containing the monomial $\overline{b}^{\overline{a}} = \prod_i b_i^{a_i}$. This polynomial can be expressed in terms of *elementary symmetric polynomials* $\sigma_i(b_1, b_2, \ldots)$ on b_i's. Let $R^{\overline{a}}(\sigma_1, \sigma_2, \ldots)$ be the respective expression. Then one defines $S^{\overline{a}}_{L-N} : \Omega^*(X) \to \Omega^{*+|\overline{a}|}(X)$ as $S^{R^{\overline{a}}}_{L-N}$. Parametrised this way, the Landweber-Novikov operations can be easily organised into the *multiplicative operation*

$$S^{Tot}_{L-N} := \sum_{\overline{a}} \overline{b}^{\overline{a}} \cdot S^{\overline{a}}_{L-N} : \Omega^*(X) \to \Omega^*(X) \otimes_{\mathbf{Z}} \mathbf{Z}[b_1, b_2, \ldots].$$

Multiplicativity property means that

$$S^{Tot}_{L-N}(x \cdot y) = S^{Tot}_{L-N}(x) \cdot S^{Tot}_{L-N}(y).$$

Specialising b_i to some values in $\mathbb{L}$ one gets the multiplicative operations $\Omega^*(X) \to \Omega^*(X)$.

Each multiplicative operation $G : A^*(X) \to B^*(X)$ provides a homomorphism of formal group laws

$$\gamma_G : (A^*, F_A(x,y)) \to (B^*, F_B(x,y)),$$

that is, the ring homomorphism $G : A^* \to B^*$ together with the (change of parameter) power series $\gamma_G(z) \in B^*[[z]]$ satisfying

$$G(F_A)(\gamma_G(x), \gamma_G(y)) = \gamma_G(F_B(x,y)).$$

Power series $\gamma_G(z)$ is just the expression of $G(c_1^A(\mathcal{L}))$ in terms of $c_1^B(\mathcal{L})$ (sufficient to know for $\mathcal{L} = \mathcal{O}(1)$ on $\mathbb{P}^\infty$). And the equation comes from the fact that:

$$\begin{aligned} G(F_A)(\gamma_G(c_1^B(\mathcal{L})), \gamma_G(c_1^B(\mathcal{M}))) = G(F_A)(G(c_1^A(\mathcal{L})), G(c_1^A(\mathcal{M}))) &= \\ G(F_A(c_1^A(\mathcal{L}), c_1^A(\mathcal{M}))) = G(c_1^A(\mathcal{L} \otimes \mathcal{M})) &= \\ \gamma_G(c_1^B(\mathcal{L} \otimes \mathcal{M})) = \gamma_G(F_B(c_1^B(\mathcal{L}), c_1^B(\mathcal{M}))) & \end{aligned}$$

For such operation to be *stable* (in certain sense) one needs the first coefficient of $\gamma_G(z)$ to be 1 ($\gamma_G(z) = z + b_1 z^2 + b_2 z^3 + \ldots$). The total Landweber-Novikov operation S^{Tot}_{L-N} is the universal multiplicative stable operation - here the coefficients $b_1, b_2, \ldots$ in the change of parameter are just independent variables.

When $R = \sigma_i$ (that is, $\overline{a} = (1, 1, \ldots, 1)$ - i-times), we will denote the respective operations $S^{\sigma_i}_{L-N}$ simply as S^i_{L-N}. One can organise S^i_{L-N} into the multiplicative operation $S^{\bullet}_{L-N} = \sum_i S^i_{L-N} : \Omega^*(X) \to \Omega^*(X)$. Clearly, this is just the specialisation of S^{Tot}_{L-N} at $b_1 = 1; b_i = 0, i \geqslant 2$.

Steenrod operations

Let $\overline{pr} : \Omega^*(X) \to \mathrm{CH}^*(X)$ be the projection. The following result is due to P. Brosnan, M. Levine and A. Merkurjev:

Theorem 0.16 *There exists (unique) operation $S^i : \mathrm{CH}^*(X)/2 \to \mathrm{CH}^{*+i}(X)/2$ called Steenrod operation making commutative the following diagram:*

$$\begin{array}{ccc} \Omega^* & \xrightarrow{S^i_{L-N}} & \Omega^{*+i} \\ {\scriptstyle\overline{pr}}\downarrow & & \downarrow{\scriptstyle\overline{pr}} \\ \mathrm{CH}^*/2 & \xrightarrow{S^i} & \mathrm{CH}^{*+i}/2. \end{array}$$

Both Steenrod and Landweber-Novikov operations commute with the pull-back morphisms.

In a similar way one can construct *reduced power operations*

$$P^i : \mathrm{CH}^*(X)/l \to \mathrm{CH}^{*+i(l-1)}(X)/l$$

corresponding to other primes l. Here one should use $S^{\overline{a}}_{L-N}$ with $\overline{a} = (l-1, l-1, \ldots, l-1)$ - i-times. In the algebro-geometric context these operations were originally constructed by V. Voevodsky in a more general situation of motivic cohomology.

Remark: Note, that if you choose some arbitrary partition $\overline{a}$ and arbitrary number l, you, in general, will not be able to find any operation $\mathrm{CH}^*/l \to \mathrm{CH}^{*+|\overline{a}|}/l$ making the respective diagram commutative.

Symmetric operations

It follows from the explicit construction of Steenrod operations by P. Brosnan that $S^i|_{\mathrm{CH}^m/2} = 0$, if $i > m$, and $S^m|_{\mathrm{CH}^m/2}$ coincide with the operation square $\square : \mathrm{CH}^m/2 \to \mathrm{CH}^{2m}/2$. It follows from the diagram above that

$$(pr \circ S^i_{L-N})(\Omega^m(X)) \subset 2 \cdot \mathrm{CH}^{m+i}(X), \text{ for } i > m, \text{ and}$$

$$(pr \circ (S^m_{L-N} - \square))(\Omega^m(X)) \subset 2 \cdot \mathrm{CH}^{2m}(X).$$

Thus, up to 2-torsion, we have well defined operations

$$\phi^{t^{i-m}} := \frac{pr \circ S^i_{L-N}}{2} : \Omega^m(X) \to \mathrm{CH}^{m+i}(X)/(2-tors.), \text{ for } i > m, \text{ and}$$
$$\phi^{t^0} := \frac{pr \circ (S^m_{L-N} - \square)}{2} : \Omega^m(X) \to \mathrm{CH}^{2m}(X)/(2-tors.).$$

In reality, this operations can be lifted to some well-defined operations

$$\Phi^{t^j} : \Omega^* \to \Omega^{2*+j}.$$

To construct such operations consider the following objects. Let $W \to X$ be some *smooth morphism* (roughly speaking, all the fibers are smooth varieties) of smooth varieties. Denote as $\square(W/X)$ the relative square $W \times_X W$; as $\widetilde{\square}(W/X)$ the Blow-up variety $Bl_{\Delta(W) \subset \square(W/X)}$, and as $\widetilde{C}^2(W/X)$ the quotient variety of $\widetilde{\square}(W/X)$ under the natural (interchanging of factors) $\mathbf{Z}/2$-action. Notice, that the locus of fixed points on $\widetilde{\square}(W/X)$ under our action will be the smooth (special) divisor of $\widetilde{\square}(W/X)$ - the preimage of the diagonal. Thus, $\widetilde{C}^2(W/X)$ will be a smooth variety. These objects fit into the diagram

$$\begin{tikzcd}
\mathbb{P}_W(T_{W/X}) \arrow[r, "j"] \arrow[d, "\varepsilon"'] & \widetilde{\square}(W/X) \arrow[r, "p"] \arrow[d, "\pi"] & \widetilde{C}^2(W/X) \arrow[d, "\xi"] \\
W \arrow[r, "\Delta"'] & \square(W/X) \arrow[r] & X.
\end{tikzcd}$$

Variety $\widetilde{C}^2(W/X)$ has natural line bundle $\mathcal{L}$ such that $p^*(\mathcal{L}) = \mathcal{O}(1)$ - the canonical line bundle of the Blow-up variety. Denote $\rho := c_1(\mathcal{L}^{-1}) \in \Omega^1(\widetilde{C}^2(W/X))$. When $X = \mathrm{Spec}(k)$, we will omit X in the respective notations: $\widetilde{\square}(W), \widetilde{C}^2(W)$. Notice, that $\widetilde{C}^2(W)$ is nothing else but $Hilb_2(W)$ - the *Hilbert scheme of the length* 2 *subschemes on* W.

Let $v : V \to X$ be the projective morphism of smooth varieties. We can decompose it in the form $V \xrightarrow{g} W \xrightarrow{f} X$, where g is a *regular embedding*, and f is *smooth projective morphism.* Then we get the following natural diagram:

$$\widetilde{C}^2(V) \overset{\alpha}{\hookrightarrow} \widetilde{C}^2(W) \overset{\beta}{\hookleftarrow} \widetilde{C}^2(W/X) \xrightarrow{\gamma} X,$$

where all the maps are projective.

Symmetric operations will be parametrised by the power series $q(t) \in \mathbb{L}[[t]]$:

$$\Phi^{q(t)} : \Omega^*(X) \to \Omega^*(X).$$

$$\Phi^{q(t)}([v : V \to X]) := \gamma_*\beta^*\alpha_*(q(\rho)) \in \Omega^*(X).$$

For given variety X we can extend symmetric operations by $\Omega^*(X)$-linearity on $q(t)$, and assume that $q(t) \in \Omega^*(X)[[t]]$.

Properties:

(0)
$$\Phi^{q(t)}(x + y) = \Phi^{q(t)}(x) + \Phi^{q(t)}(y) + q(0)xy$$

In particular $\Phi^{q(t)}$ is linear if $q(0) = 0$.

(1) $\Phi^{q(t)}$ commutes with the pull-back morphisms;

(2) If $f : X \hookrightarrow Y$ is a regular embedding with normal bundle $\mathcal{N}_f$, and $q(t) \in \Omega^*(Y)[[t]]$. Then

$$\Phi^{q(t)}(f_*(x)) = f_*\Phi^{f^*(q(t))\cdot c^{\Omega}_{\bullet}(\mathcal{N}_f)(t)}(x),$$

where $c_{\bullet}^{\Omega}(\mathcal{V})(t) = \prod_i(\lambda_i -_{\Omega} t)$, where $\lambda_i \in \Omega^1$ are *roots* of $\mathcal{V}$, and $-_{\Omega}$ is the subtraction in the sense of the universal formal group law.

(3) $\Phi^{q(t)}$ is trivial on the classes of embeddings. Really, if $v : V \to X$ is an embedding, we can take $W = X$, and then the variety $\widetilde{C}^2(W/X)$ will be empty. Thus, the symmetric operations provide the obstructions for the cobordism class to be represented by the embedding.

(4) $2 \cdot (pr \circ \Phi^{t^r})|_{\Omega^m} = (-1)^r \cdot (pr \circ S_{L-N}^{r+m})$. Thus, with the help of the symmetric operations one can get cycles twice as small as with the help of the Landweber-Novikov operations. This difference can be crucial if one works with the varieties where the effects related to prime 2 are important (like quadrics, for example).

Remark: Actually, the properties $(0) - (3)$ determine the operations $\Phi^{q(t)}$ uniquely up to renormalisation $q(t) \mapsto q(t) \cdot r(t)$, for fixed $r(t) \in \mathbb{L}[[t]]$ satisfying $r(0) = 1$.

The most interesting symmetric operations are not expressible in terms of the Landweber-Novikov operations, and cannot be organised into the multiplicative operation. Nevertheless, some of them are, and these operations are related to the *Steenrod operations in Cobordism theory.*

Lecture 6

u-invariants of fields.

In this lecture we will demonstrate the applications of the technique discussed earlier to the u-invariants of fields.

Let k be a field. Define the u-invariant of k as

$$u(k) := \max(\dim(q)|q \;-\; \text{anisotropic form over } k).$$

Examples:

(1) k-algebraically closed, then $u(k) = 1$;

(2) $u(\mathbb{R}) = \infty$;

(3) k-finite, then $u(k) = 2$;

(4) k-local, then $u(k) = 4$;

(5) k-global, then $u(k) = \begin{cases} \infty, & \text{if there are real embeddings } k \subset \mathbb{R}; \\ 4, & \text{otherwise} \end{cases}$.

(6) $k = F[[t_1, \dots, t_n]]$, where F-algebraically closed, then $u(k) = 2^n$.

So, in a certain sense, the u-invariant gives some idea how far our field is from being algebraically closed (of course, it can see only one of the projections of such a distance).

The natural question arises: what are the possible values of this invariant?

It is easy to see that $u(k)$ cannot take values $3, 5$, and 7.

Example: Let us show that $u(k) \neq 3$. Really, if $u(k)$ would be 3, then all the forms of dimension $\geqslant 4$ over k would be isotropic, and some form of dimension 3 would be anisotropic. Up to a scalar, such form is $\langle 1, -a, -b\rangle$ and the respective projective quadric is conic $C_{\{a,b\}}$. But as we saw in Lecture 3, such conic is isotropic if and only if the respective 2-dimensional 2-fold Pfister quadric $Q_{\{a,b\}}$ is (for example, because $Q_{\{a,b\}} = C_{\{a,b\}} \times C_{\{a,b\}}$). This gives a contradiction, since $\dim(\langle\langle a, b\rangle\rangle) = 4$.

"Conjecture" of Kaplansky (1953) predicted that the only possible values are powers of two.

It was disproved by A. Merkurjev (1989), who constructed fields with all even u-invariants. Further disproved by O. Izhboldin (1999), who constructed the field k with $u(k) = 9$ - the first odd value (> 1).

The basic ingredient of the construction is the

Merkurjev tower of fields

Let F be some field, and $M \in \mathbb{N}$. We want to construct some extension of F, where all forms of dimension $> M$ will be isotropic. Suppose we have just one form q. There are many extensions of F making q isotropic. For example $\overline{F}$ - the algebraic closure of F. But we want the one which would behave in a most gentle way with respect to everything. Such field is, of course, $F(Q)$ - the generic point of the quadric Q - any other field making q isotropic will be a specialisation of this one. If we want to make two forms q_1, q_2 isotropic, then we should use the field $F(Q_1 \times Q_2)$, etc. ... .

Denote as J the set of all forms of dimension $> M$ over F, and define the new field F' by the formula

$$F' := \varinjlim_{I \subset J} F(\times_{i \in I} Q_i),$$

where I runs over all finite subsets of J. This field has the property, that any form q of dimension $> M$ defined over F is isotropic over F'. And it is universal one among the extensions E/F with such property.

Starting with some field k, consider the sequence of fields

$$k = k_0 \hookrightarrow k_1 \hookrightarrow k_2 \hookrightarrow \dots,$$

where $k_{i+1} := (k_i)'$. Denote $k_\infty := \lim_{\to i} k_i$. Then all the forms of dimension $> M$ defined over k_∞ are isotropic (since any such form is defined on some finite level k_i, and thus, becomes isotropic over k_{i+1}). In other words, $u(k_\infty) \leqslant M$. But we would want the equality. For this we need some anisotropic form p of dimension M over k_∞. Better to have it already over k, and then check that it stays anisotropic over k_∞. Of course, to be able to control this, one needs to know something interesting about p (not just the fact that it is anisotropic). Formalising, we need a form p of dimension M over k, and two properties A and B on the set of field extensions E/k, where

$$A(E) \text{ is satisfied } \Leftrightarrow \ p|_E \text{ is anisotropic,}$$

and A and B satisfy the following axioms:

(1) $B \Rightarrow A$;

(2) $B(k)$ is satisfied;

(3) $B(F)$ is satisfied, $\dim(q) > M \Rightarrow B(F(Q))$ is satisfied;

(4) $B(F_j)$, for directed system of fields is satisfied $\Rightarrow B(\lim_{\to_j} F_j)$ is satisfied.

In this case, $A(k_\infty)$ is satisfied, and $u(k_\infty) = M$.

So, we need only to choose the form p and the right property B.

Choice of Merkurjev:

To each quadratic form p one can assign its *Clifford algebra* $C(q)$ defined as $T_k(V_p)/(v^2 - p(v), \forall v \in V_p)$ - the quotient of the tensor algebra of the underlying vector space by the explicit relations. This algebra has a natural $\mathbf{Z}/2$-grading, and it "is not far" from being a central simple algebra. We will be interested only in the case, where $p \in I^2$, that is, $\dim(p)$ is even and $\det_\pm(p) = 1$. In such a case, $C(p) = Mat_{2\times 2}(k) \otimes_k C'(p)$, where C' is a central simple algebra over k. In the case $M = 2n$ - even, Merkurjev have chosen the following property:

$$B(E) \text{ is satisfied } \Leftrightarrow \ C'(p|_E) \text{ is a division algebra.}$$

One starts with the *generic* quadratic form of dimension $2n$ from I^2 - that is, the form $\langle a_1, \dots, a_{2n-1}, (-1)^n \prod_{i=1}^{2n-1} a_i \rangle$ over the field $k = F(a_1, \dots, a_{2n-1})$. Let us check the axioms:

1) $B \Rightarrow A$, since $p = \mathbb{H} \perp r \Rightarrow C'(p) = Mat_{2\times 2}(k) \otimes_k C'(r)$.

2) $B(k)$ is satisfied, since the C' of the generic form as above is division.

4) Clear, since zero divisors are defined on the finite level.

3) This is the only nontrivial part. The proof here is based on the *Index reduction formula of Merkurjev.* This formula describes what happens to the index of the central simple algebra over the generic point of a quadric. It says that the index of the division algebra C over $k(Q)$ can drop at most by the factor 2, and the latter happens if and only if there is a k-algebra homomorphism $C_0(q) \to C$, where $C_0(q)$ is the *even Clifford algebra of* q (the degree zero part of $C(q)$).

Notice, that $C_0(q)$ is either a simple algebra, or a product of two isomorphic simple algebras, and if $\dim(p) = 2n$ is even, and $\dim(q) > \dim(p)$, then the size of each simple factor in $C_0(q)$ will be bigger than the size of $C'(p)$, so we do not have maps $C_0(q) \to C'(p)$. Thus, the condition (3) is fulfilled, and $u(k_\infty) = 2n$.

<u>Another choice for even M</u>

Let us make another choice of the property B. We will choose one based on the EDI - the elementary discrete invariant of quadrics (see Lecture 4). Namely, we start with the generic form $p = \langle a_1, \dots, a_{2n} \rangle$ over $k = F(a_1, \dots, a_{2n})$, and the property:

$$B(E) \text{ is satisfied } \Leftrightarrow y_{d,0}(p|_E) \text{ is not defined over } k,$$

where $d = [\dim(P)/2] = n - 1$.

In other words, $EDI(p|_E)$ should have the form

$$\begin{array}{ccccc} \circ & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ? & ? & ? & \cdots & ? \end{array}$$

1) $B \Rightarrow A$ since $A(E)$ is satisfied if and only if $y_{0,0}$ is not defined over E

(we remind, that $y_{0,0}$ is just the class of a rational point on $P|_{\overline{E}}$), and $y_{i,j}$ is defined implies $y_{l,j}$ is defined for any $l > i$.

2) $B(k)$ is satisfied, since EDI of the generic form is empty.

4) Follows, since for any X/k, $CH^*(X|_{\lim_{\to j} F_j}) = \lim_{\to j} CH^*(X|_{F_j})$ (with any coefficients).

3) This is again the only nontrivial part, and it follows from the following:

Theorem 0.17 *Let Y be smooth quasi-projective variety over some field k of characteristic zero. Let Q be smooth projective quadric over k, and $\overline{y} \in \mathrm{CH}^m(Y|_{\overline{k}})/2$ be some element. Suppose $2m < \dim(Q)$. Then*

$$\overline{y} \text{ is defined over } k \ \Leftrightarrow\ \overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q).$$

Indeed, one just needs to take $\overline{y} = y_{d,0}$. Then $m = \dim(P) - d = d < \dim(Q)/2$ for any q bigger than p, and the Theorem implies what we need.

Shortly, we succeeded by controlling not the class $y_{0,0}$, but the smaller codimensional (!) class $y_{d,0}$. The point, of course, is: the smaller is the codimension of the cycle, the easier it is to control its rationality.

Notice, that the bound $2m < \dim(Q)$ is optimal: for any pair $\dim(Q), m$ not satisfying the inequality, one can find variety Y, cycle $\overline{y}$, and a quadric Q of needed codimension and dimension, so that $\overline{y}|_{k(Q)}$ is defined over $k(Q)$, but $\overline{y}$ is not defined over k. Just take Q generic, and $\overline{y} = y_{d,0} \times pt$ on $G(d, Q) \times \mathbb{P}^{m-d}$.

The proof of the above Theorem uses the *Symmetric operations* in Algebraic Cobordism (see Lecture 5). If $\overline{y}|_{\overline{k(Q)}}$ is defined over $k(Q)$, then we lift the respective element first to $\mathrm{CH}^*(Y \times Q)/2$, and, finally, to $\Omega^*(Y \times Q)$ using the natural surjections:

$$\mathrm{CH}^*(Y|_{k(Q)})/2 \twoheadleftarrow \mathrm{CH}^*(Y \times Q)/2 \twoheadleftarrow \Omega^*(Y \times Q).$$

Then we restrict it to $Y \times Q_s$ for the subquadrics $e_s : Q_s \to Q$ of different dimensions, and apply the composition of the various symmetric operations with the projection $(\pi_s)_*$, after which we map the results to $\mathrm{CH}^*(Y)/2$, and add them with certain coefficients.

$$\begin{tikzcd}
\mathrm{CH}^*(Y|_{k(Q)})/2 & \mathrm{CH}^*(Y \times Q)/2 \arrow[l, two heads] & \Omega^*(Y \times Q) \arrow[l, two heads] \arrow[r, "e_s^*"] & \Omega^*(Y \times Q_s) \arrow[d, "(\pi_s)_*"] \\
& \mathrm{CH}^*(Y)/2 \arrow[u] & & \Omega^*(Y) \arrow[ll]
\end{tikzcd}$$

It appears, that if $2m < \dim(Q)$, one can choose the coefficients in such a way that the result will be independent of all the choices we made, and will be equal to $\overline{y}$ when restricted to $\overline{k}$.

Let me demonstrate the usefulness of our Theorem on the following:

Example: *EDI* of a Pfister forms. Let $\alpha \in \mathrm{K}_n^M(k)/2$ be nonzero pure symbol, and $q_\alpha = \langle\langle \alpha \rangle\rangle$ be the respective anisotropic Pfister form. Then in $EDI(Q_\alpha)$ the marked points will be the ones strictly above the Main (NW-SE) diagonal. Indeed, consider $Q = Q_\alpha$, $Y = G(i, Q_\alpha)$, $\overline{y} = y_{i,j} \in \mathrm{CH}^{\dim(Q_\alpha)-i-j}(Y)/2$. Since $Q_\alpha|_{k(Q_\alpha)}$ is completely split, all elementary classes on Q_α are defined over this field. But then, by the Theorem, those ones which are of sufficiently small codimension, i.e., exactly the ones living strictly above the Main diagonal, are defined already over the base field k. It remains to see that the other ones are not defined. Because of the rule
$$\begin{tikzcd} \bullet & \bullet \\ & \bullet \arrow[ul] \arrow[u] \end{tikzcd}$$
it is sufficient to check that the NW-corner ($y_{d,0}$) is not defined over k. But if it would be defined, all the elementary classes $y_{d,j}$ on the last Grassmannian $G(d, Q_\alpha)$ would be defined. But the product $\prod_{j=0}^{d} y_{d,j}$ of these classes is equal to the class of rational point on $G(d, Q_\alpha)|_{\overline{k}}$. So, this would imply that Q_α is completely split (we use the Theorem of Springer here, claiming that the quadric has a rational point, if it has one of odd degree). Thus, $y_{d,0}$ is not defined over k, and $EDI(Q_\alpha)$ is as we described:

$$\begin{array}{cccccc}
\circ & \bullet & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \circ & \cdots & \bullet & \bullet \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\circ & \circ & \circ & \cdots & \circ & \bullet \\
\circ & \circ & \circ & \cdots & \circ & \circ
\end{array}$$

u-invariants $2^r + 1$, $r \geqslant 3$

The same ideas can be used to construct the fields with some odd u-invariants. These values are $2^r + 1$, $r \geqslant 3$. In the case of u-invariant 9 we get method different from that of O.Izhboldin, and for $r > 3$ we get the values not known before.

For odd dimensional form p we cannot use the class $y_{d,0}$ anymore, since for $q = p \perp \langle \det_{\pm}(p) \rangle$, the class $y_{d,0}(p|_{k(Q)})$ will always be defined, although $\dim(q) > \dim(p)$. So, our condition should involve somehow the classes $y_{i,0}$ $i < d$, since these are the only ones which are defined as soon as P has a rational point.

We have the following:

Theorem 0.18 *Let* $\dim(p) = 2^r + 1$*,* $r \geqslant 3$*, and* $EDI(P)$ *looks as*

$$\begin{array}{ccccc} ? & \circ & \circ & \cdots & \circ \\ \circ & \circ & \circ & \cdots & \circ \\ \circ & \circ & \circ & \cdots & \circ \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \circ & \circ & \circ & \cdots & \circ \end{array}$$

Suppose $\dim(q) > \dim(p)$*. Then* $EDI(P|_{k(Q)})$ *has the same property.*

The above theorem immediately implies that the property:

$$B(E) \text{ is satisfied } \Leftrightarrow \ EDI(p|_E) \text{ is as above}$$

satisfies the axiom (3). Let us take the generic form p of dimension $2^r + 1$, then all the other axioms will be readily fulfilled as well, and $u(k_\infty) = 2^r + 1$.

The proof of Theorem 0.18 uses certain extensions of Theorem 0.17, the knowledge of action of the Steenrod operations on the elementary classes and the fact that on the last Grassmannian the subring of k-rational cycles is always generated by the k-rational elementary classes. So, it involves a bit more than the case of even u-invariants.

In the end, let me mention some useful literature:

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